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1975 J. Phys. A: Math. Gen. 8 1159

(http://iopscience.iop.org/0305-4470/8/7/019)

J. Phys. A: Math. Gen., Vol. 8, No. 7, 1975. Printed in Great Britain. © 1975

The approach to space charge limited current flow between concentric spheres

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Received 20 January 1975

Abstract. Poisson's equation in spherical symmetry is solved numerically in terms of the three following variables: the anode current, the cathode electric field and the ratio of the anode radius r_1 to the cathode radius r_0, where $r_1 > r_0$. The treatment, which is non-relativistic, is carried out for $1 \leqslant r_1/r_0 \leqslant 10^6$ and for values of cathode field ranging from zero to 99·9 % of the space charge free value. A simple empirical relation is found connecting the anode current and the cathode field which is valid over a wide range of field values and is completely independent of radius for $10^2 < r_1/r_0 < 10^6$.

1. Introduction

Many vacuum emission devices, such as photocells, thermionic valves and electron guns, have an electrode geometry that approximates to parallel planes, coaxial cylinders or concentric spheres. If the cathode emits an infinite supply of electrons of zero energy, the fully space charge limited current in these geometries can be evaluated following the analyses of Child (1911), Langmuir (1913), Langmuir and Blodgett (1923, 1924) and Langmuir and Compton (1931). These analyses assume that the field at the cathode surface is reduced to zero by space charge effects. However, if the supply of electrons is a function of the cathode field, such as by field emission, the fully limited current can be several orders of magnitude greater than the true emission current. Evaluation of the true current requires simultaneous solution of the Fowler–Nordheim field emission relation and the diode cathode field/anode current characteristic. Recently Wheeler (1975) studied the approach to space charge limitation for coaxial cylinders and showed that the problem could be completely specified in terms of three dimensionless parameters defining the cathode field, the anode current and the electrode radii. This paper applies the same techniques to concentric spheres. The Fowler–Nordheim relation is extremely sensitive to small changes in cathode field; consequently the calculations here are carried out for fields ranging from zero up to 99·9 % of the space charge free value. Furthermore a range of 10^6 in anode/cathode radii is covered which is sufficient to embrace the finest point emitters used in field emission microscopy.

2. Mathematical formulation

Consider a concentric spherical geometry comprising an outer anode sphere of radius r_1 maintained at a potential V_1 with respect to the inner cathode sphere of radius r_0. In the steady state the potential and space charge density ρ in the region $r_0 \leqslant r \leqslant r_1$ are related through Poisson's equation:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -4\pi\rho. \tag{1}$$

If the electrons are emitted from the cathode with zero energy then ρ can be related to the total current I and also to the electron velocity v which, in turn, can be related to the local potential:

$$I = -4\pi r^2\rho v, \qquad \tfrac{1}{2}mv^2 = eV. \tag{2}$$

This equation assumes that the potentials are sufficiently low to warrant a non-relativistic treatment and that the current I is sufficiently low for self-magnetic fields not to influence the electron motion. Equations (1) and (2) combine to give

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}V}{\mathrm{d}r}\right) = \left(\frac{m}{2e}\right)^{1/2} IV^{-1/2}. \tag{3}$$

It is convenient to express I in terms of the current I_{p} where

$$\frac{I_{\mathrm{p}}}{4\pi r_1^2} = \frac{1}{9\pi}\left(\frac{2e}{m}\right)^{1/2} V_1^{3/2} r_1^{-2}. \tag{4}$$

Thus the right-hand side of this equation is the Child–Langmuir limiting current density that would flow between infinite parallel planes separated by a distance r_1 and at a potential difference V_1. Now redefine the position and potential variables in the following non-dimensional manner:

$$x = \ln(r/r_0), \qquad y = \left(\frac{9}{8}\frac{I_{\mathrm{p}}}{I}\right)^{2/3}\frac{V}{V_1}. \tag{5}$$

Equation (3) then reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}y^{-1/2}. \tag{6}$$

Multiplying throughout by $2\mathrm{e}^{2x}\,\mathrm{d}y/\mathrm{d}x$ and integrating over $x \geqslant 0$ ($r \geqslant r_0$, $V \geqslant 0$, $y \geqslant 0$),

$$\mathrm{e}^{2x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - A^2 = \int_0^y \mathrm{e}^{2x}y^{-1/2}\,\mathrm{d}y \tag{7}$$

where $A = (\mathrm{d}y/\mathrm{d}x)_0$. Equation (5) shows that this parameter is proportional to the electric field E at the cathode surface; it is instructive to relate this field to the field E_0 that would exist at the cathode surface in the absence of space charge effects. Then

$$A = (E/E_0)y_1/(1-\mathrm{e}^{-x_1}), \qquad \text{where } E_0 = V_1/r_0(1-\mathrm{e}^{-x_1}). \tag{8}$$

$A = 0$ ($E = 0$) corresponds to fully space charge limited current flow, while $A = \infty$ ($E = E_0$) corresponds to zero current and so to the absence of space charge effects. This latter limit follows from equation (5) which requires $y_1 = \infty$ at $I = 0$. Square-rooting equation (7), inverting and integrating over $0 \leqslant x \leqslant x_1$ ($r_0 \leqslant r \leqslant r_1$, $0 \leqslant V \leqslant V_1$, $0 \leqslant y \leqslant y_1$) gives

$$x_1 = \int_0^{y_1} \mathrm{e}^x\left(A^2 + \int_0^y \mathrm{e}^{2x}y^{-1/2}\,\mathrm{d}y\right)^{-1/2}\mathrm{d}y. \tag{9}$$

For small values of x_1, such that $e^{x_1} \simeq 1$, this equation can be integrated to give

$$x_1 = (A^3/3)[(1+2y_1^{1/2}/A^2)^{3/2} - 3(1+2y_1^{1/2}/A^2)^{1/2} + 2] \qquad \text{for } x_1 < 1. \quad (10)$$

This equation is identical to the corresponding equation in cylindrical geometry (Wheeler 1975) with the corresponding definitions of x, y and A. Substitution from equations (5) and (8) and rearrangement enables equation (10) to be expressed as a function of the two parameters E/E_0 and $x_1^2 I/I_p$:

$$\begin{aligned}(x_1^2 I/I_p)^{1/2} = {} & [1+\tfrac{9}{16}(E/E_0)^2(x_1^2 I/I_p)^{-1}]^{3/2} \\ & -\tfrac{27}{16}(E/E_0)^2(x_1^2 I/I_p)^{-1}[1+\tfrac{9}{16}(E/E_0)^2(x_1^2 I/I_p)^{-1}]^{1/2} \\ & +2[\tfrac{9}{16}(E/E_0)^2(x_1^2 I/I_p)^{-1}]^{3/2} \qquad \text{for } x_1 < 1. \qquad (11)\end{aligned}$$

In the cylindrical case the result was the same except that $x_1^2 I/I_p$ was replaced by $x_1^2 e^{-x_1} I/I_p$. As $x_1 \to 0$ we have from equations (4), (5) and (8):

$$\left.\begin{aligned} x_1^{-2} I_p/4\pi r_1^2 &\to \frac{1}{9\pi}\left(\frac{2e}{m}\right)^{1/2} V_1^{3/2} d^{-2} \\ E_0 &\to V_1/d \end{aligned}\right\} \qquad \text{as } x_1 \to 0$$

where $r_1 - r_0 = d$. Consequently the two parameters in equation (11) respectively reduce to the ratio of the cathode field to the space charge free value and the ratio of the current to the Child–Langmuir limit, both for a plane geometry.

3. Results and discussion

Equation (9) was integrated numerically by Simpson's rule employing an iterative technique that used equation (10) as a first approximation. The integration was performed for $0 \leqslant x_1 \leqslant 13{\cdot}9$ ($1 \leqslant r_1/r_0 \leqslant 10^6$) and for ten values of A in the range $0 \leqslant A \leqslant 200$, yielding ten (x_1, y_1) relations which were transformed to ten $(r_1/r_0, I/I_p, E/E_0)$ relations using equations (5) and (8). This enabled ten pairs of coordinates $(r_1/r_0, I/I_p)$ to be obtained for any specified value of E/E_0 in the range $0 \leqslant E/E_0 \leqslant 0{\cdot}999$. Figures 1(*a*) and 1(*b*) show the results obtained for $0 \leqslant E/E_0 \leqslant 0{\cdot}99$ and for $1 \leqslant r_1/r_0 \leqslant 10^2$ where, in the region $r_1/r_0 < 1{\cdot}1$, the curves are expressed analytically by equation (11). Figures 1(*c*) and 1(*d*) show the behaviour in the region $10^2 \leqslant r_1/r_0 \leqslant 10^6$. In the case of $E/E_0 = 0$ the values of I/I_p correspond to the factor α^{-2} tabulated by Langmuir and Blodgett (1924) and comparison shows agreement to better than 2% over the entire range of r_1/r_0 presented. In figures 1(*c*) and 1(*d*) it is seen that the curves are uniformly displaced along the ordinate from each other and this behaviour is also exhibited for values of E/E_0 up to the maximum value of 0·999 used in these calculations. This indicates that for large r_1/r_0 there is a unique relation between E/E_0 and I_E/I_0 which is independent of radius, where I_E is the current appropriate to E/E_0 and I_0 that appropriate to $E/E_0 = 0$. The average values of I_E/I_0 over the range $10^2 \leqslant r_1/r_0 \leqslant 10^6$ are shown as a function of E/E_0 in figure 2, where each coordinate corresponds to one curve in figures 1(*c*) and 1(*d*) with an additional six coordinates from computations in the range $0{\cdot}994 \leqslant E/E_0 \leqslant 0{\cdot}999$. For $E/E_0 \geqslant 0{\cdot}6$ the coordinates in this double logarithmic

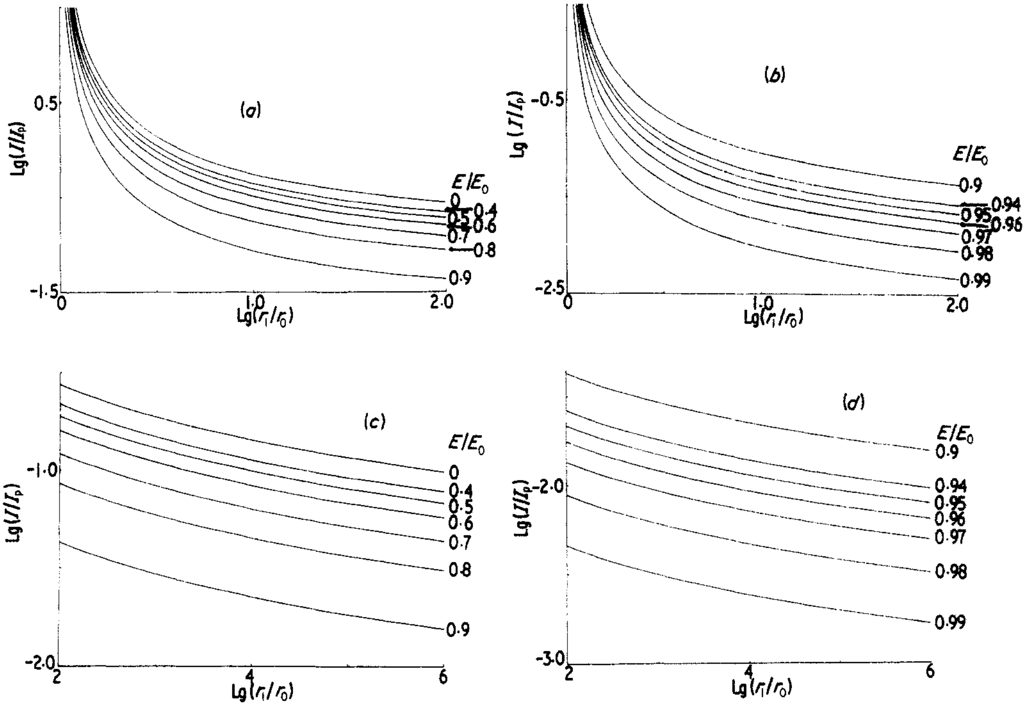

Figure 1. Dependence of anode current on electrode radii for selected values of cathode field: (*a*) $0 \leqslant r_1/r_0 \leqslant 10^2$, $0 \leqslant E/E_0 \leqslant 0{\cdot}9$; (*b*) $0 \leqslant r_1/r_0 \leqslant 10^2$, $0{\cdot}9 \leqslant E/E_0 \leqslant 0{\cdot}99$; (*c*) $10^2 \leqslant r_1/r_0 \leqslant 10^6$, $0 \leqslant E/E_0 \leqslant 0{\cdot}9$; (*d*) $10^2 \leqslant r_1/r_0 \leqslant 10^6$, $0{\cdot}9 \leqslant E/E_0 \leqslant 0{\cdot}99$.

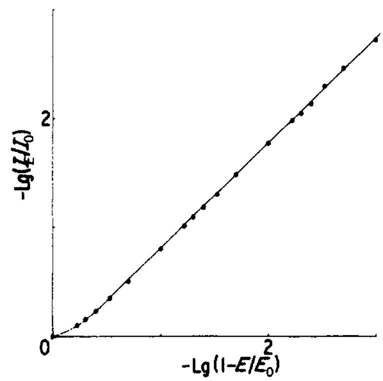

Figure 2. Dependence of anode current on cathode field valid for $10^2 \leqslant r_1/r_0 \leqslant 10^6$.

plot lie very close to the full straight line shown, defined by

$$I_E/I_0 = 1{\cdot}445[1-(E/E_0)]^{0{\cdot}967} \qquad \text{for} \quad \begin{array}{l} 10^2 \leqslant r_1/r_0 \leqslant 10^6 \\ 0{\cdot}6 \leqslant E/E_0 \leqslant 0{\cdot}999. \end{array}$$

The scatter of the coordinates about the line is equivalent to a maximum variation in

I_E/I_0 of $\pm 5\%$. This scatter is of a random nature and arises predominantly from cumulative errors in the calculations.

In applying these calculations to the evaluation of field emission currents it may be necessary to correct for the fact that the Fowler–Nordheim relation assumes a plane cathode boundary. A theoretical treatment of field emission from a spherical surface by Sodha and Kaw (1968) shows that the plane assumption is valid, provided that the radius is greater than about 10 Å. For smaller radii the emission exceeds that of the plane assumption.

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